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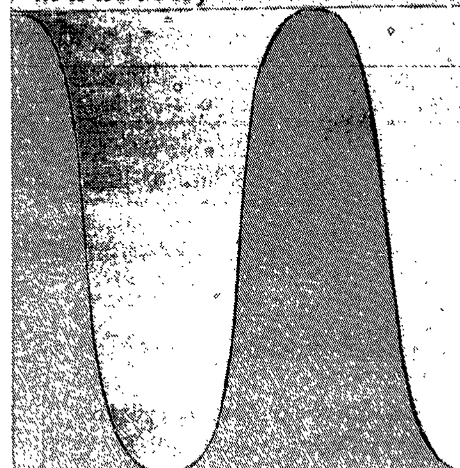

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ON A DIFFERENCE-INTEGRAL EQUATION

Part II. Laplace Transform Treatment

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1. Introduction. Consider the difference-integral equation

$$\varphi(t) - \varphi(t+1) = \int_{-1}^{1} K(s)\varphi(t-s)ds \quad , \quad 0 \leqq t < \infty \quad , \tag{1.1}$$

where the given kernel $K(s)$, $-1 \leqq s \leqq 1$, and the solution $\varphi(t)$, $-1 \leqq t < \infty$, are assumed to be (real or) complex continuous functions. We are interested particularly in the asymptotic behavior as $t \to \infty$ of an arbitrary solution $\varphi(t)$ of (1.1).

This equation was treated by the authors in [1] with the aid of concepts and methods of functional analysis. The same equation is treated below with the aid of the Laplace transform. Since we shall make no smoothness assumptions on $\varphi(t)$, such as local bounded variation, we cannot apply directly the complex inversion formula for the Laplace transform. Nevertheless, we manage to get much the same results which a standard approach yields under suitable smoothness conditions.

Quite general classes of difference-differential, difference-integral and difference-differential-integral equations are treated in the book by Pinney [4]. He uses the Laplace transform along with smoothness and other assumptions. Many references to the literature are given in Pinney's book.

This paper is not entirely independent of [1]. Certain results obtained there

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essentially by means of classical analysis are fundamental to both approaches to the problem. According to Theorem 2 of [1], each continuous function defined for $-1 \leqq t \leqq 1$ which satisfies (1.1) at $t = 0$ has a unique continuous extension $\varphi(t)$, $-1 \leqq t < \infty$, which satisfies (1.1) for all $t \geqq 0$. Thus, each solution $\varphi(t)$ of (1.1) is determined by its values for $-1 \leqq t \leqq 1$. Furthermore, by (3.15), (3.4) and (2.8) of [1],

$$|\varphi(t)| \leqq e^{ct} \max_{-1 \leqq s \leqq 1} |\varphi(s)| \quad , \quad 0 \leqq t \leqq \infty \quad , \tag{1.2}$$

where

$$c = \ln[(1 + Me^{M})(1 + 2M)] \quad , \quad M = \max_{-1 \leqq s \leqq 1} |K(s)| \quad . \tag{1.3}$$

2. <u>The Laplace transform of $\varphi(t)$</u>. Henceforth, let $\varphi(t)$ denote an arbitrary but fixed solution of the difference-integral equation. The Laplace transform of $\varphi(t)$, $0 \leqq t < \infty$, is

$$\hat{\varphi}(z) = \int_0^\infty e^{-zt} \varphi(t) dt \qquad (z = x + iy) \tag{2.1}$$

at least for $x > c$, where c is defined by (1.3).

Multiply both members of equation (1.1) by e^{-zt} and integrate to obtain

$$\hat{\varphi}(z) = \frac{A(z)}{\Psi(z)} \quad , \tag{2.2}$$

where

$$A(z) = \int_{-1}^{1} e^{-z(s+t)} K(s) \varphi(t) dt ds - e^{z} \int_{-1}^{1} e^{-zt} \varphi(t) dt \quad , \tag{2.3}$$

$$\Psi(z) = 1 - e^{z} - \int_{-1}^{1} e^{-zs} K(s) ds \quad . \tag{2.4}$$

Let (2.3) and (2.4) define $A(z)$ and $\Psi(z)$ for all z. Let (2.2) define $\hat{\varphi}(z)$ as a meromorphic function; its poles comprise a (perhaps proper) subset of the zeros of $\Psi(z)$.

Note that $\Psi(z)$ is independent of $\varphi(t)$ and that $A(z)$ depends only on the values of $\varphi(t)$ for $-1 \leqq t \leqq 1$. Hence, $\hat{\varphi}(z)$ depends on $\varphi(t)$ only for $-1 \leqq t \leqq 1$, which agrees with the fact that $\varphi(t)$ is determined by its values for $-1 \leqq t \leqq 1$.

The function $\Psi(z)$ was introduced in another connection in [1], where it was called the <u>characteristic function</u>. For later purposes we outline below some of its properties. For further details see Section 6 of [1].

A function $\varphi(t) = e^{z_0 t}$ satisfies (1.1) if and only if $\Psi(z_0) = 0$. More generally, the functions $\varphi_j(t) = t^j e^{z_0 t}$, $0 \leqq j < J$, satisfy (1.1) if and only if z_0 is a zero of $\Psi(z)$ of order J. The zeros of $\Psi(z)$ are countably infinite in number. If $\Psi(z_0) = 0$ then $\mathrm{Re}(z_0) \leqq c$, where c is defined by (1.3). For some positive integer m, there exist <u>simple</u> zeros z_n, $n = \pm m, \pm(m+1), \ldots$, of $\Psi(z)$ such that

$$z_n - 2n\pi i \to 0 \quad \text{as} \quad |n| \to \infty \ .$$

Excluding these points, $\Psi(z)$ has only a finite number of other zeros z with $\mathrm{Re}(z) \geqq \alpha$, where α is any real number.

Finally, let us express $\Psi(z)$ in the form

$$\Psi(z) = 1 - e^z - G(z) \ , \tag{2.5}$$

where

$$G(z) = \int_{-1}^{1} e^{-zs} K(s)\,ds \ . \tag{2.6}$$

By the Riemann-Lebesgue lemma,

$$G(z) \to 0 \quad \text{as} \quad |y| \to \infty \qquad (z = x + iy) \tag{2.7}$$

uniformly on each finite x-interval.

Also for future reference, we state some of the properties of $A(z)$. It follows easily from (2.3) that

$$A(z) = \int_{-1}^{1} e^{-zt} h(t)\,dt \ , \tag{2.8}$$

where

$$h(t) = \begin{cases} -\int_{-1}^{t} K(s)\varphi(t-s)\,ds - \varphi(t+1) \ , & -1 \leqq t < 0 \ , \\ \\ \int_{t}^{1} K(s)\varphi(t-s)\,ds \ , & 0 \leqq t \leqq 1 \ . \end{cases} \tag{2.9}$$

By (1.1) with $t = 0$,

$$h(0) - h(0-) = \varphi(0) \ . \tag{2.10}$$

Finally, by (2.8) and the Riemann-Lebesgue lemma,

$$A(z) \to 0 \quad \text{as} \quad |y| \to \infty \qquad (z = x + iy) \tag{2.11}$$

uniformly on each finite x-interval.

It follows from (1.2) by a standard argument involving the Plancherel theorem (cf. [5], pg. 80) that

$$\varphi(t) = \underset{\eta \to \infty}{\text{l.i.m.}} \frac{1}{2\pi i} \int_{\xi - i\eta}^{\xi + i\eta} \hat{\varphi}(z) e^{zt}\,dz \ , \quad \xi > c \ , \tag{2.12}$$

where l.i.m. indicates limit in the mean and is the limit in the norm of $L_2(0 \leqq t < \infty)$. We shall make only limited use of this result. Under suitable smoothness conditions on $\varphi(t)$, such as local bounded variation, (2.12) holds

On the interval $1 \leqq t < \infty$, $p(t)$ is periodic with period one. It follows from (2.9) and (2.10) that $p(t)$ is continuous. Furthermore, $p(t)$ is obviously bounded. The following result is needed for later purposes.

LEMMA 3.1. For each $\gamma < 0$,

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \hat{p}(z)e^{zt}dz = 0 \;, \qquad t > 1 \;, \tag{3.3}$$

where the improper Riemann integral converges uniformly to zero on each finite t-interval.

Proof. Fix $\gamma < 0$. Choose t_1, t_2 and ϵ arbitrarily such that $1 < t_1 < t_2$ and $\epsilon > 0$. We must prove that

$$\left| \int_{\gamma-i\eta_1}^{\gamma+i\eta_2} \hat{p}(z)e^{zt}\,dz \right| < \epsilon \;, \qquad t_1 \leqq t \leqq t_2 \;, \tag{3.4}$$

for all sufficiently large η_1 and η_2. In view of (3.1), $\hat{p}(z)$ is regular in the left half-plane. Therefore, if $\alpha < \beta < \gamma$, then

$$\int_{\gamma-i\eta_1}^{\gamma+i\eta_2} \hat{p}(z)e^{zt}dz =$$

$$\left\{ \int_{\gamma-i\eta_1}^{\beta-i\eta_1} + \int_{\beta-i\eta_1}^{\alpha-i\eta_1} + \int_{\alpha-i\eta_1}^{\alpha+i\eta_2} + \int_{\alpha+i\eta_2}^{\beta+i\eta_2} + \int_{\beta+i\eta_2}^{\gamma+i\eta_2} \right\} \hat{p}(z)e^{zt}dz \;.$$

First, we choose $\beta < \gamma$ such that

$$\left| \int_{\alpha+i\eta}^{\beta+i\eta} \hat{p}(z)e^{zt}dz \right| < \frac{\epsilon}{5} \;, \qquad \begin{cases} t_1 \leqq t \leqq t_2 \;, \\ -\infty < \alpha < \beta \;, \\ -\infty < \eta < \infty \;. \end{cases}$$

This is possible because, by (3.1) and (2.8),

$$\hat{p}(z) = O(e^{-x}) \text{ as } x \to -\infty \qquad (z = x + iy)$$

uniformly for $-\infty < y < \infty$. By (2.11), if η_1 and η_2 are sufficiently large, then

$$\Big|\int_{\gamma-i\eta_1}^{\beta-i\eta_1} \hat{p}(z)e^{zt}dz\Big| < \frac{\epsilon}{5}, \qquad t_1 \leqq t \leqq t_2 ,$$

$$\Big|\int_{\beta+i\eta_2}^{\gamma+i\eta_2} \hat{p}(z)e^{zt}dz\Big| < \frac{\epsilon}{5}, \qquad t_1 \leqq t \leqq t_2 .$$

For each such choice of η_1 and η_2, we choose α so large negatively that

$$\Big|\int_{\alpha-i\eta_1}^{\alpha+i\eta_2} \hat{p}(z)e^{zt}dz\Big| < \frac{\epsilon}{5}, \qquad t_1 \leqq t \leqq t_2 .$$

The foregoing inequalities imply (3.4) and, hence, the assertion of the lemma.

4. <u>The function $q(t)$.</u> Define $q(t)$ such that

$$\varphi(t) = p(t) + q(t), \qquad 0 \leqq t < \infty . \tag{4.1}$$

By (2.13) and (3.1),

$$\hat{q}(z) = \frac{\hat{\varphi}(z)G(z)}{1 - e^{z}} . \tag{4.2}$$

If z is a pole of $\hat{q}(z)$ then either z is a pole of $\hat{\varphi}(z)$ or $z = 2\pi ni$ for some integer n. Thus, the poles $z = x + iy$ with $x \neq 0$ of $\hat{q}(z)$ and $\hat{\varphi}(z)$ coincide.

<u>LEMMA 4.1.</u> If $\hat{\varphi}(z)$ is regular for $\operatorname{Re}(z) = \alpha$, where $\alpha \neq 0$, then the improper integral

$$q_\alpha(t) = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} \hat{q}(z)e^{zt}dz, \qquad -\infty < t < \infty , \tag{4.3}$$

converges uniformly on each finite t-interval. Furthermore,

$$q_\alpha(t) = e^{\alpha t}Q_\alpha(t), \qquad -\infty < t < \infty , \tag{4.4}$$

where $Q_a(t)$ is bounded and uniformly continuous. If $\hat{\varphi}(z)$ is regular for $\mathrm{Re}(z) \geqq \xi$, where $\xi > 0$, then

$$(4.5) \qquad q(t) = q_\xi(t) = O(e^{\xi t}) \;, \qquad t \geqq 0 \;.$$

Proof. Using (4.2), we express (4.3) in the form

$$q_a(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\varphi}(a + iy) \frac{G(a + iy)}{1 - e^{a+iy}} e^{iyt} dy \;.$$

By (2.2) and (2.8),

$$\hat{\varphi}(a + iy) = \frac{A(a + iy)}{\Psi(a + iy)} \;,$$

where $A(a + iy)$, considered as a function of y with a fixed, is the Fourier transform of a function of t in $L_1(-\infty, \infty) \cap L_2(-\infty, \infty)$. Therefore, $A(a + iy)$ is a continuous function of y and, by the Plancherel theorem, $A(a + iy) \in L_2(-\infty, \infty)$. It follows from (2.5) and (2.7) that $1/\Psi(a + iy)$ is bounded for $|y|$ sufficiently large. Since, by hypothesis, $\hat{\varphi}(z)$ is regular for $z = a + iy$, $-\infty < y < \infty$, the foregoing results imply that

$$\hat{\varphi}(a + iy) \in L_2(-\infty, \infty) \;.$$

By a similar argument,

$$\frac{G(a + iy)}{1 - e^{a+iy}} \in L_2(-\infty, \infty) \;.$$

Since the product of two functions in $L_2(-\infty, \infty)$ lies in $L_1(-\infty, \infty)$ (cf. the Schwarz inequality),

$$\varphi(a + iy) \frac{G(a + iy)}{1 - e^{a+iy}} \in L_1(-\infty, \infty) \;,$$

which implies the assertions of the theorem regarding (4.3) and (4.4). The other

assertions follow by means of a standard theorem.*

LEMMA 4.2. If $\hat{\varphi}(z)$ is regular for $\mathrm{Re}(z) = \alpha$, where $\alpha < 0$, then

$$q_\alpha(t) = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} \hat{\varphi}(z)e^{zt}dz \ , \qquad t > 1 \ , \tag{4.6}$$

where the integral converges uniformly on each finite t-interval. For $t > 2$, $q_\alpha(t)$ satisfies the difference-integral equation (1.1).

Proof. Lemmas 3.1 and 4.1 and the equation $\hat{\varphi}(z) = \hat{p}(z) + \hat{q}(z)$ yield (4.6) and the uniform convergence. It follows from (4.6) and (2.2) that

$$q_\alpha(t) - q_\alpha(t+1) - \int_{-1}^{1} K(s)q_\alpha(t-s)ds = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} A(z)e^{zt}dz \ , \qquad t > 2 \ .$$

A procedure similar to that used in the proof of Lemma 3.1 shows that the last integral vanishes.

5. The asymptotic behavior of $\varphi(t)$. Some of the principal results of this paper are collected here.

THEOREM 5.1. If $\hat{\varphi}(z)$ is regular for $\mathrm{Re}(z) \geqq \xi$, where $\xi \neq 0$, then

$$\varphi(t) = O(e^{\xi t}) \ . \tag{5.1}$$

Moreover, if $\xi < 0$, then

$$\varphi(t) = q_\xi(t) = \frac{1}{2\pi i}\int_{\xi-i\infty}^{\xi+i\infty} \hat{\varphi}(z)e^{zt}dz \ , \qquad t > 1 \ , \tag{5.2}$$

where the integral converges uniformly on each finite t-interval.

Proof. According to (4.1), $\varphi(t) = p(t) + q(t)$, where $p(t)$ is bounded. By

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* THEOREM (cf. Doetsch [7], pg.107, Th. 6). If the transform $\hat{f}(z)$ of $f(t)$ is regular for $\mathrm{Re}(z) \geqq \xi$ and if the complex inversion integral converges uniformly to $g(t)$ on each finite t-interval with $t \geqq T$, then $f(t) = g(t)$ for $t \geqq T$.

(4.4), $q(t) = O(e^{\xi t})$. Therefore, (5.1) holds if $\xi > 0$. If $\xi < 0$, then (5.1) follows from Lemma 4.2 and the standard theorem cited earlier (cf. [7], pg. 107).

In some cases, the asymptotic estimate (5.1) can be improved with the aid of the residue theorem.

THEOREM 5.2. Suppose that $\hat{\varphi}(z)$ is regular for $\mathrm{Re}(z) = a$, where $a > 0$ and that $\hat{\varphi}(z)$ has exactly n poles z_k, $k = 1,\dots,n$, with $\mathrm{Re}(z_k) > a$. Then $\varphi(t)$ has the form

$$(5.3) \qquad \varphi(t) = \sum_{k=1}^{n} P_k(t)e^{z_k t} + q_a(t) + p(t) \ , \qquad t \geqq 0 \ ,$$

where $P_k(t)$ is a polynomial of degree one less than the order of the pole z_k of $\hat{\varphi}(z)$. The sum in (5.3) is dominant asymptotically as $t \to \infty$.

Proof. It follows from Lemma 4.1 that

$$(5.4) \qquad q(t) = q_a(t) + \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} [\hat{q}(z)e^{zt}] \ .$$

Since $\varphi(t) = p(t) + q(t)$ and the residue of $\hat{q}(z)e^{zt}$ is the inverse transform of the principal part of the Laurent expansion of $\hat{q}(z)$ about z_k, the assertions of the theorem are correct.

THEOREM 5.3. Suppose that $\hat{\varphi}(z)$ is regular for $\mathrm{Re}(z) \geqq a$, where $a < 0$, and that $\hat{\varphi}(z)$ has exactly n poles z_k, $k = 1,\dots,n$, with $\mathrm{Re}(z_k) > a$. Then $\varphi(t)$ has the form

$$(5.5) \qquad \varphi(t) = \sum_{k=1}^{n} P_k(t)e^{z_k t} + q_a(t) \ , \qquad t > 1 \ ,$$

where $P_k(t)$ is a polynomial of degree one less than the order of the pole z_k

of $\varphi(z)$. Each of the $n+1$ terms in (5.5) satisfies the difference-integral equation (1.1). The sum in (5.5) is dominant asymptotically as $t \to \infty$.

Proof. The residue of $\varphi(z)e^{zt}$ at z_k has the form

$$r_k(t) = P_k(t)e^{z_k t} \quad , \quad k = 1, \ldots, n \ .$$

It is easy to verify that each $r_k(t)$ satisfies (1.1). By (5.2), with $\varphi(t)$ replaced by $\varphi(t) - \sum_{k=1}^{n} r_k(t)$, and (4.6),

$$\varphi(t) - \sum_{k=1}^{n} r_k(t) = q_a(t) - \sum_{k=1}^{n} \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} r_k(z)e^{zt}dz$$

for each $t > 1$. By a routine argument,

$$\frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} r_k(z)e^{zt}dz = 0 \quad , \quad k = 1, \ldots, n \ .$$

Thus, the theorem is proved.

6. Further properties of $\varphi(t)$. Although the results derived below do not yield uniform asymptotic estimates for $\varphi(t)$ as $t \to \infty$, they are interesting in themselves. The next theorem is in the direction of a Heaviside expansion for $\varphi(t)$.

THEOREM 6.1. If $\hat{\varphi}(z)$ is regular for $\mathrm{Re}(z) = a$, where $a < 0$, then

$$\varphi(t) = \underset{N \to \infty}{\mathrm{l.i.m.}} \sum_{\substack{x > a \\ |y| < (2N+1)\pi}} \mathrm{Res}[\varphi(z)e^{zt}] + q_a(t) \quad , \quad t > 1 \ , \tag{6.1}$$

where the limit is in the mean square on each finite t-interval. For each sufficiently large positive integer m there exist simple poles z_n and complex numbers a_n, $n = \pm m, \pm(m+1), \ldots,$ such that

(6.2) $$z_n - 2n\pi i \to 0 \quad \text{as} \quad n \to \infty$$

and

(6.3) $$\varphi(t) = \sum_{n=\pm m}^{\pm\infty} a_n e^{z_n t} + \sum_{\substack{x > \alpha \\ |y| < (2m-1)\pi}} \operatorname{Res}[\hat{\varphi}(z)e^{zt}] + q_\alpha(t), \quad t > 1 ,$$

where the series converges in the mean square on each finite t-interval.

Proof. Equation (6.1) follows by routine arguments from (2.12), (4.6) and (2.11). Relations (6.2) and (6.3) follow from (6.1) and the remarks in Section 2 on the zeros of $\Psi(z)$. Further details are omitted.

Suppose that $\hat{\varphi}(z)$ is regular for $\operatorname{Re}(z) \geqq 0$. In view of Theorem 5.1, it seems reasonable to conjecture that $\varphi(t) \to 0$ or, at least, that $\varphi(t)$ remains bounded as $t \to \infty$. Rudin [8] has addressed himself to these conjectures. He has constructed an example in which $\varphi(t)$ is unbounded. Although the kernel $K(s)$ in his example has a possible discontinuity at $s = 0$, the construction seems flexible enough to permit elimination of the discontinuity.

Although we are not able to assert that $\varphi(t)$ is bounded if $\hat{\varphi}(z)$ is regular for $\operatorname{Re}(z) \geqq 0$, the following theorem shows that $\varphi(t)$ tends to zero in a certain mean square sense as $t \to \infty$.

THEOREM 6.2. If $\hat{\varphi}(z)$ is regular for $\operatorname{Re}(z) \geqq 0$, then

(6.4) $$\int_S^{S+\delta} |\varphi(t)|^2 dt \to 0 \quad \text{as} \quad S \to \infty , \qquad \delta > 0 .$$

Proof. By hypothesis, all the poles of $\hat{\varphi}(z)$ have negative real parts. Therefore, (6.4) holds with $\varphi(t)$ replaced by the second sum in (6.3). Furthermore, (6.4) holds with $\varphi(t)$ replaced by $q_\alpha(t)$ with $\alpha < 0$, since $q_\alpha(t) = 0(e^{\alpha t})$. It remains to consider the first term in (6.3), which we denote by

(6.5) $$f(t) = \sum_{n=\pm m}^{\pm\infty}{}' a_n e^{z_n t} .$$

The series converges in the mean square on each finite t-interval. In view of (6.2), we can assume without loss of generality that

(6.6) $$|z_n - 2n\pi i| < 2 \log 2 , \quad |n| \geqq m .$$

In [9], Anselone considered sums of the type (6.5). It was proved that (6.4) holds with $\varphi(t)$ replaced by $f(t)$. The assertion of the theorem now follows by means of (6.3) and the triangle inequality.

REFERENCES

[1] Anselone, P. M. and Bueckner, H. F., On a Difference-Integral Equation, MRC Technical Summary Report #171, August 1960, University of Wisconsin, Madison, Wisconsin.

[2] Taylor, W. C., and Treuenfels, P. M., The Emptying of a Gun Tube, Ballistic Research Laboratories Report No. 976 (SECRET), March 1956, Aberdeen Proving Ground, U. S. Army.

[3] Giese, J. H., On the Asymptotic Behavior of Certain One-Dimensional Flows, Ballistic Research Laboratories Report No. 1098, March 1960, Aberdeen Proving Ground, U. S. Army.

[4] Pinney, E., Ordinary Difference-Differential Equations, 1959, University of California Press, Berkeley and Los Angeles.

[5] Widder, D. V., The Laplace Transform, 1946, Princeton University Press.

[6] Churchill, R. V., Operational Mathematics, 2nd ed., 1958, McGraw-Hill Book Company, Inc., New York.

[7] Doetsch, G., Theorie und Anwendung der Laplace-Transformation, 1937, Springer-Verlag, Berlin.

[8] Rudin, W., The Influence of Damping Factors, MRC Technical Summary Report #220, March, 1961, University of Wisconsin, Madison, Wisconsin.

[9] Anselone, P. M., On the Asymptotic Behavior of the Sum of a "Non-Harmonic Fourier series", MRC Technical Summary Report #291, January 1962, University of Wisconsin, Madison, Wisconsin.